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# NOTES ON MARTINGALE THEORY

J. L. DOOB
UNIVERSITY OF ILLINOIS

## 1. Introduction

Although several writers, for example Bernstein, Lévy, and Ville, had used what would now be identified as martingale concepts, the first systematic studies appeared in [2] and [3]. Since then, martingale theory has been applied extensively, but little progress has been made in the theory itself. The purpose of the present paper is to point out how much spade work remains to be done in the theory, by deriving new theorems without the use of deep technical apparatus.

Throughout this paper, the more appropriate nomenclature "submartingale," "supermartingale" is used, rather than the "semimartingale," "lower semimartingale" found in [3]. The unifying thread in the following work will be the fact that certain simple operations on submartingales transform them into submartingales. This leads to a new submartingale convergence theorem, to a sharpening of the upcrossing inequality, and thereby into an examination of apparently hitherto unnoticed interrelations between martingale and potential theory.

## 2. A new submartingale convergence theorem

The theorems about derivatives of set functions on nets led Chow [1] to study submartingales relative to atomic fields, and he deduced a new submartingale convergence theorem. Theorem 2.1 generalizes Chow's result, and shows how it can be made to depend on the fact that certain transformations take submartingales into submartingales.

We first prove a lemma. The point of this lemma is that, although the sample sequences of a submartingale increase on the average, they may increase more than is necessary to preserve the submartingale property. It may therefore be possible to cut down the random variables of a submartingale the first time a given barrier is passed, and thereby to obtain a submartingale which is bounded from above. The condition (2.1) of the lemma is unusual in that, instead of restricting the excess of the sample sequence the first time the barrier is crossed, as is customary, the probability of crossing is supposed not too small.

LEMMA 2.1. *Let* $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ *be a submartingale, and let* $\delta$, $a$, $b$ *be specified constants, with* $a \leqq b$, $\delta > 0$. *Suppose that* $\Lambda_0 = \emptyset$, *and, if* $m \geqq 1$, *suppose that, almost everywhere on some set* $\Lambda_m$ *in* $\mathfrak{F}_m$,

$$
(2.1)\qquad \begin{aligned} & x_j < b, && j = 1, \cdots, m, \\ & P\{x_{m+1}(\omega) \geqq b|\mathfrak{F}_m\} \geqq \delta, && P\{x_{m+1}(\omega) < a|\mathfrak{F}_m\} = 0. \end{aligned}
$$

*Let $\nu(\omega)(\leqq\infty)$ be the minimum integer $k$ with $x_k(\omega) \geqq b$, and define*

$$
(2.2)\qquad \begin{aligned} x_n'(\omega) &= x_n(\omega) \quad \text{if} \quad n < \nu(\omega), \\ &\left.\begin{aligned} &= \alpha && \text{on} \quad \Lambda_m \\ &= x_{m+1}(\omega) && \text{off} \quad \Lambda_m \end{aligned}\right\} \text{if} \quad n \geqq m + 1 = \nu(\omega), \end{aligned}
$$

*where $\alpha$ is any constant $\geqq (b - a + a\delta)/\delta$. Then $\{x_n', \mathfrak{F}_n, n \geqq 1\}$ is a submartingale.*

We can and shall suppose that $a = 0$, replacing $x_n$, $b$, $c$ respectively by $x_n - a$, $b - a$, $c - a$ otherwise. If $\Lambda \in \mathfrak{F}_m$,

$$
(2.3)\qquad \int_{\Lambda\{\nu(\omega) \leqq m\}} x_m'\, dP = \int_{\Lambda\{\nu(\omega) \leqq m\}} x_{m+1}'\, dP
$$

because the integrands are identical. Let $M_m = \Lambda\Lambda_m\{\nu(\omega) > m\}$ and let $N_m$ be the remainder of $\Lambda\{\nu(\omega) > m\}$. Then

$$
(2.4)\qquad P\{\mathrm{M}_m, \nu(\omega) = m + 1\} = \int_{\mathrm{M}_m} P\{x_{n+1}(\omega) \geqq b|\mathfrak{F}_m\}\, dP \geqq \delta P\{\mathrm{M}_m\}.
$$

Hence

$$
(2.5)\qquad \int_{\mathrm{M}_m} x_m'\, dP = \int_{\mathrm{M}_m} x_m\, dP \leqq bP\{\mathrm{M}_m\} \leqq \frac{b}{\delta} P\{\mathrm{M}_m, \nu(\omega) = m + 1\} \leqq \int_{\mathrm{M}_m} x_{m+1}'\, dP.
$$

Furthermore, using the submartingale inequality,

$$
(2.6)\qquad \int_{\mathrm{N}_m} x_m'\, dP = \int_{\mathrm{N}_m} x_m\, dP \leqq \int_{\mathrm{N}_m} x_{m+1}\, dP = \int_{\mathrm{N}_m} x_{m+1}'\, dP.
$$

Adding (2.3), (2.5), and (2.6), we find that the $x_n'$ process is a submartingale.

We remark that the lemma is still true, with essentially no change in proof if in the third line of (2.2) $x_n'(\omega)$ is set equal to $x_n(\omega)$.

THEOREM 2.1. *Let $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ be a submartingale, and let $\delta$, $b$, $c$ be specified constants, with $\delta > 0$, $b \leqq c$. Suppose that, for each value of $n \geqq 1$, at least one of the following conditions is satisfied, almost everywhere where* $\max_{j \leqq n} x_j < b$.

$$
(2.7)\qquad P\{x_{n+1}(\omega) \geqq b|\mathfrak{F}_n\} \geqq \delta, \qquad P\{x_{n+1}(\omega) \geqq c|\mathfrak{F}_n\} = 0.
$$

*Then* $\lim_{n\to\infty} x_n$ *exists and is finite almost everywhere where* $\sup_n x_n < b$.

In proving the theorem it is a slight convenience to assume that $x_1 < b$, and we shall do so. If the inequality is not already satisfied, simply adjoin $x_0 = \min[x_1, b]$ to the sequence. To prove the theorem, choose $a < b$, and define $x_{an} = \max[a, x_n]$. Then the lemma is applicable to the $x_{an}$ process, if $\Lambda_m$ is the set where $\max_{j \leqq m} x_j < b$ and where simultaneously the first inequality in (2.7) is true, yielding a submartingale $\{x_{an}', n \geqq 1\}$, and clearly $x_{an}' \leqq \max[\alpha, c]$.

Hence $\lim_{n\to\infty} x'_{an}$ exists and is finite with probability 1, by a standard submartingale convergence theorem. We shall suppose below that $\alpha$ is chosen to be the smallest stated admissible value in the application of the lemma, and that $-a$ is so large that $\alpha \geqq c$. In view of the leeway in the choice of $a$, it follows that, almost everywhere where $\sup_n x_n < b$, either $\lim_{n\to\infty} x_n$ exists and is finite or $\inf_n x_n = -\infty$. Let $\Lambda$ be the $\omega$-set where $\sup_n x_n < b$ and simultaneously $\inf_n x_n = -\infty$. We shall show that the $P\{\Lambda\} = 0$. In fact, if $\mathrm{M} \in \mathfrak{F}_k$, and if $\max_{j\leqq k} x_j < b$ on M,

$$(2.8) \qquad \int_{\mathrm{M}} x'_{ak}\, dP \leqq \int_{\mathrm{M}} \lim_{n\to\infty} x'_{an}\, dP \leqq aP\{\Lambda\mathrm{M}\} + \frac{b - a + a\delta}{\delta} P\{\tilde{\Lambda}\mathrm{M}\}$$

$$\leqq a\left[P\{\Lambda\mathrm{M}\} - \frac{1-\delta}{\delta} P\{\tilde{\Lambda}\mathrm{M}\}\right] + \left|\frac{b}{\delta}\right|.$$

Here $\tilde{\Lambda}$ is the complement of $\Lambda$. Now M can be chosen arbitrarily close to $\Lambda$ in the usual measure metric of sets, at the expense of choosing $k$ large. Hence we can suppose that M has been chosen to make the bracket in (2.8) strictly positive. But then, when $a \to -\infty$, the right side of (2.8) becomes $-\infty$, whereas the left side becomes the integral of $x_k$ over M, and this contradiction means that $P\{\Lambda\} = 0$, as was to be proved.

As an application of the theorem, suppose that $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ is a submartingale with respect to the indicated fields and that each field $\mathfrak{F}_n$ is atomic. Suppose further that $\inf P\{\mathrm{M}_1\}/P\{\mathrm{M}_2\} = K > 0$, where $\mathrm{M}_1$ ranges through the atoms of $\mathfrak{F}_2, \mathfrak{F}_3, \cdots$, and, if $\mathrm{M}_1$ is an atom of $\mathfrak{F}_n$, $\mathrm{M}_2$ is the atom of $\mathfrak{F}_{n-1}$ containing it. Then, for every $b$, condition (2.7) is satisfied almost everywhere, with $\delta = K$, $c = b$. Hence $\lim_{n\to\infty} x_n$ exists and is finite almost everywhere where $\sup_n x_n < \infty$ in this case. This case of the theorem was proved (using a different method) by Chow [1]. (His measure space is an arbitrary $\sigma$-finite measure space, and his random variables may not have finite integrals but the extension to this context involves no difficulty.)

## 3. The upcrossing inequality

The technical results of martingale theory are to a considerable extent based on elementary inequalities. These in turn again depend, or can easily be made to depend, essentially on the fact that a submartingale goes into a submartingale under optional sampling. To illustrate the basic character of this invariance under optional sampling, we derive the fundamental upcrossing inequality (due to the author for martingales and to Snell for submartingales) in a new and slightly stronger form, using the invariance, as suggested by Hunt. (The proof is in principle of course not very different from that in [3].) In the following we write $x^+$ for $\max [x, 0]$.

THEOREM 3.1. *Let $\{x_j, \mathfrak{F}_j, j \leqq n\}$ be a submartingale, and let $\beta$ be the number of upcrossings by $(x_1, \cdots, x_n)$ of the interval $[r_1, r_2]$. Then*

$$E\{\beta|\mathfrak{F}_1\} \leqq \frac{E\{(x_n - r_1)^+|\mathfrak{F}_1\} - (x_1 - r_1)^+}{r_2 - r_1} \tag{3.1}$$

*with probability* 1.

To prove this inequality we can and shall suppose that $x_j \geqq r_1$ for all $j$, replacing $x_j$ by max $[x_j, r_1]$ if necessary to achieve this. Now define successively: $\nu_1 = 1$; $\nu_2$ is the smallest value of $j \geqq 1$ for which $x_j = r_1$; $\nu_3$ is the smallest value of $j \geqq \nu_2$ for which $x_j \geqq r_2$, and so on, alternating in going to $r_1$ and above $r_2$. If $\nu_j$ is not defined by the above prescription, define $\nu_j = n$. Then certainly $\nu_n = n$, and

$$x_n - x_1 = \sum_{j=2}^{n} (x_{\nu_j} - x_{\nu_{j-1}}). \tag{3.2}$$

The first $\beta$ summands with odd $j$ are at least $r_2 - r_1$. The next summand with odd $j$ is $\geqq 0$, and all later ones vanish. Hence

$$x_n - x_1 \geqq \sum{}' (x_{\nu_j} - x_{\nu_{j-1}}) + \beta(r_2 - r_1), \tag{3.3}$$

where the primed summation symbol means that the index $j$ is to be even. Now $x_{\nu_1}, \cdots, x_{\nu_n}$ is a submartingale, since it is obtained by optional sampling from a submartingale. Hence each summand in (3.3) has a positive conditional expectation relative to $\mathfrak{F}_1$, so that

$$E\{x_n - x_1|\mathfrak{F}_1\} \geqq E\{\beta|\mathfrak{F}_1\}(r_2 - r_1) \tag{3.4}$$

with probability 1, as was to be proved.

In most applications, the integrated form

$$E\{\beta\} \leqq \frac{E\{(x_n - r_1)^+\} - E\{(x_1 - r_1)^+\}}{r_2 - r_1} \tag{3.1$'$}$$

is all that is needed, and in fact the second expectation in the numerator of (3.1′) is usually not needed. The inequality can be sharpened slightly by replacing $x_1$ and $x_{\nu_1}$ in the proof by $x_{\nu_2}$. This leads to

$$E\{\beta\} \leqq \frac{1}{r_2 - r_1} \int_{\Lambda} (x_n - r_1)^+ \, dP, \qquad \Lambda = \{\min [x_1, \cdots, x_{n-1}] \leqq r_1\}. \tag{3.1$''$}$$

Let $\beta'$ be the number of downcrossings of $[r_1, r_2]$ by $x_1, \cdots, x_n$. Then $\beta' \leqq \beta + 1$ and hence (3.1) can be used to limit $\beta'$. Alternatively, one can modify the above proof as follows. Define $\nu_1 = 1$; $\nu_2$ is now the smallest value of $j \geqq 1$ for which $x_j \geqq r_2$; $\nu_3$ is the smallest value of $j \geqq \nu_2$ for which $x_j = r_1$, and so on, with $\nu_j = n$ if $\nu_j$ is not otherwise defined. Then (3.2) is true. The first $\beta'$ summands with odd $j$ are $\leqq -(r_2 - r_1)$. The next summand with odd $j$ is $\leqq (x_n - r_2)^+$. Hence (3.3) now has the form

$$x_n - x_1 \leqq \sum{}' (x_{\nu_j} - x_{\nu_{j-1}}) - \beta'(r_2 - r_1) + (x_n - r_2)^+. \tag{3.3$'$}$$

The conditional expectation of the primed sum relative to $\mathfrak{F}_1$ is at most that of $x_n - x_1$, so that

$$E\{\beta'|\mathfrak{F}_1\} \leqq \frac{E\{x_n - r_2)^+|\mathfrak{F}_1\}}{r_2 - r_1}. \tag{3.5}$$

If $y_1, \cdots, y_n$ is a positive supermartingale, a common hypothesis in potential-theoretic studies, and if $\gamma$ is the number of downcrossings by $y_1, \cdots, y_n$ of the interval $[s_1, s_2]$, $0 \leqq s_1 < s_2$, the obvious application of (3.1) with $x_j = -y_j$ leads to

$$E\{\gamma|\mathfrak{F}_1\} \leqq \frac{E\{(s_2 - y_n)^+|\mathfrak{F}_1\} - (s_2 - y_1)^+}{s_2 - s_1} \leqq \frac{\min [y_1, s_2]}{s_2 - s_1}. \tag{3.6}$$

This inequality was used by Hunt [4].

## 4. Martingales and potential theory

In some ways the most natural application of martingale theory is to potential theory. Without attempting to justify the not entirely outrageous assertion that one reasonable generalization of a subharmonic function is a submartingale, we remark that there is a correspondence between theorems of potential theory and those of martingale theory not only due to the fact that a large class of potential functions considered on certain stochastic process paths define supermartingales, but also due to the fact that many of the methods used in one theory have counterparts in the other. There is thus a natural interplay between potential-theoretic and probabilistic techniques in which sometimes one and sometimes the other is more useful. For example the generalized Fatou theorem that positive superharmonic functions on a Green space have limits in a suitable sense at almost all (harmonic measure) Martin boundary points was proved probabilistically before it was proved by potential-theoretic methods. As an elementary but important example of the interplay between the two theories we consider the downcrossing inequality (3.6) for positive supermartingales. We shall derive a dual inequality and the potential-theoretic counterpart of both inequalities.

If $v$ is any function defined on an $N$-dimensional Green space, say on an open connected set of $N$-space whose complement has strictly positive capacity, define $\bar{v}(\xi) = \inf_w w(\xi)$, where $w$ ranges through those positive superharmonic functions which are $\geqq v$ on the space. (We shall suppose that the class of functions $w$ is not empty. This condition is trivially satisfied in the applications to be made.) The function $\bar{v}$ need not be superharmonic, but, according to standard theorems of potential theory, there is a superharmonic $\tilde{v} \leqq \bar{v}$ for which there is equality except on a set of zero capacity. If $v_1 = v_2$ except on a set of zero capacity, then $\tilde{v}_1 = \tilde{v}_2$. If $u$ is a function on the Green space, if $C$ is a subset of the space, and if $v$ is defined as $u$ on $C$ and 0 otherwise, then we write $u_C$ and $[u_C]$ for $\bar{v}$ and $\tilde{v}$ respectively. In particular, if $u$ is itself positive and superharmonic, $[u_C] \leqq u$, and there is equality at every fine limit point of $C$, that is, at every point which is a limit point of $C$ in the Cartan fine topology. It follows from the above that $[[u_A]_B] = [(u_A)_B]$, and we write this function simply as $[u_{AB}]$, using

the corresponding notation for further iterations of this operation. We write $1_C$ for $u_C$ if $u$ is identically 1.

In probability language, if $C$ is an analytic set $[1_C](\xi)$ is known to be the probability that a Brownian path from $\xi$ meets $C$ at some strictly positive time. More generally, it can be shown that, if $A$ and $B$ are analytic sets, $[1_{AB}](\xi)$ is the probability that a Brownian path from $\xi$ hits a point of $B$ at some strictly positive time and then goes on to hit a point of $A$ at some strictly later time. There is a similar probability interpretation of $[1_{ABC}]$, and so on.

The probability counterpart of the operation taking $v$ into $\tilde{v}$ is the following. Let $\{x_t, t \in T\}$ be a stochastic process, and let $\{\mathfrak{F}_t, t \in T\}$ be a monotone increasing family of Borel fields of measurable sets such that $x_t$ is measurable $\mathfrak{F}_t$. If $s \in T$, define $\tilde{x}_s$ as the infimum (generalized infimum neglecting sets of probability 0) of all $y_s$; here $y_s$ is to be a random variable from a positive supermartingale $\{y_t, \mathfrak{F}_t, t \in T\}$ with $y_t \geqq x_t$ with probability 1 for each $t$. (We shall suppose that the class of $y_t$ supermartingales is not empty. This condition is trivially satisfied in the applications to be made.) We can choose $\tilde{x}_s$ to be measurable $\mathfrak{F}_s$. The $\tilde{x}_t$ process is a positive supermartingale relative to the given family of fields, and, for each $t$, $\tilde{x}_t \geqq x_t$ with probability 1. Limiting operations of this type were first considered by Snell [5].

This limit operation is simpler to analyze when $T$ is finite. Suppose then that $\{x_j, \mathfrak{F}_j, j \leqq n\}$ is a given stochastic process, where $\mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_n$ are Borel fields of measurable sets and $x_j$ is measurable $\mathfrak{F}_j$. It is no restriction for our purposes to suppose that the random variables are positive and we shall do so, assuming also that their expectations are finite. Then $\tilde{x}_n = x_n$ with probability 1; otherwise $\tilde{x}_n$ could be decreased, contrary to the minimal property. For $j < n$, according to the side condition on the minimizing procedure and to the supermartingale property,

$$\tilde{x}_j \geqq \max\,[x_j, E\{\tilde{x}_{j+1}|\mathfrak{F}_j\}], \tag{4.1}$$

with probability 1, and in fact there must be equality with probability 1 because the right side defines successively for $j = n-1, n-2, \cdots$ random variables determining a supermartingale dominating the $x_j$ process.

We shall now define the probabilistic counterpart of the potential-theoretic operation taking $u$ into $[u_A]$. The counterpart of the domain of $u$ appears to be a specified set of Borel fields. We shall therefore assume as given a probability measure space and monotone sequence $\mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_n$ of Borel fields of its measurable subsets. Let $\Lambda_j$ be a set in $\mathfrak{F}_j$ and let $x_j$ be a positive random variable, measurable $\mathfrak{F}_j$, with finite expectation, $j \leqq n$. Define $y_j$ as $x_j$ on $\Lambda_j$ and as 0 otherwise. Then the operation taking $x_j$ into $\tilde{y}_j$ is the counterpart of that taking $u$ into $[u_A]$.

In particular, if $\{x_j, \mathfrak{F}_j, j \leqq n\}$ is a positive supermartingale, the algorithm for $\tilde{y}_j$ yields

$$\tilde{y}_j = E\{\phi_{jj}x_j + \cdots + \phi_{jn}x_n|\mathfrak{F}_j\}, \tag{4.2}$$

where $\phi_{jj}$ is the indicator function of $\Lambda_j$ and, for $k > j$, $\phi_{jk}$ is the indicator function of the set of points in $\Lambda_k$ but not in any $\Lambda_i$ with $j \leqq i < k$. (We have now found the probability version of the classical sweeping out process of potential theory.) If each $x_j$ is identically 1, (4.2) reduces to

$$\tilde{y}_j = P\{\Lambda_j \cup \cdots \cup \Lambda_n | \mathfrak{F}_j\}. \tag{4.3}$$

In particular, if the $x_j$ process is a specified reference supermartingale, if $A$ is a specified linear Borel set, and if $\Lambda_j = \{x_j(\omega) \in A\}$, we shall write $x_A(j)$ for $\tilde{y}_j$ as defined by (4.2), in analogy with the potential-theoretic counterpart discussed above. Then $1_A(1)$ is the conditional probability relative to $\mathfrak{F}_1$ that some $x_j$ has a value in $A$, $(1_A)_B(1)$, which we write $1_{AB}(1)$, is the conditional probability relative to $\mathfrak{F}_1$ that some $x_j$ has a value in $B$ and that some $x_k$ with $k \geqq j$ has a value in $A$, and so on. We conclude that, if $\gamma$ is the number of times an $x_j$ process sample sequence passes from $B$ to $A$,

$$E\{\gamma | \mathfrak{F}_1\} = 1_{AB}(1) + 1_{ABAB}(1) + \cdots \tag{4.4}$$

with probability 1.

THEOREM 4.1. *Let $u$ be a positive superharmonic function on a Green space. Suppose that $0 \leqq s_1 < s_2$ and define the sets $A$, $B$ by*

$$A = \{\xi : u(\xi) \leqq s_1\}, \qquad B = \{\xi : u(\xi) \geqq s_2\}. \tag{4.5}$$

*Then*

$$1_{AB} + 1_{ABAB} + \cdots \leqq \frac{\min [u, s_2]}{s_2 - s_1} \tag{4.6}$$

*and*

$$1_{BA} + 1_{BABA} + \cdots \leqq \frac{\min [u, s_1]}{s_2 - s_1}. \tag{4.7}$$

According to our discussion in this section, the left sides of (4.6) and (4.7) at a point $\xi$ are respectively equal except possibly on a set of zero capacity to the expected number of downcrossings and upcrossings of $[s_1, s_2]$ by $u$ on Brownian paths from $\xi$. We shall give potential-theoretic proofs rather than reducing the theorem to probability crossing inequalities, however. In the following we assume that $u \leqq s_2$, replacing $u$ by $\min [u, s_2]$ if necessary to achieve this. To prove (4.6) we first note that $1_{AB} \leqq 1_B \leqq u/s_2$ and that $u_A/s_1 \leqq 1_A$. Hence

$$1_{ABA} \leqq 1_{BA} \leqq \frac{s_1}{s_2} 1_A, \tag{4.8}$$

from which we deduce that $1_{ABAB} \leqq (s_1/s_2)1_{AB}$. Iterating this inequality and summing we find that the left side of (4.6) is at most $s_2 1_{AB}/(s_2 - s_1) \leqq u/(s_2 - s_1)$, as was to be proved. To prove (4.7) we note that, from (4.8),

$$1_{BABA} \leqq \frac{s_1}{s_2} 1_{ABA} \leqq \frac{s_1}{s_2} 1_{BA}. \tag{4.9}$$

Iterating and summing we find that the left side of (3.7) is at most

$$\frac{s_2 1_{BA}}{s_2 - s_1} \leqq \frac{u_A}{s_2 - s_1} \leqq \frac{\min [u, s_1]}{s_2 - s_1} \tag{4.10}$$

as was to be proved.

This theorem and proof are valid even if we suppose of $u$ merely that it is the infimum of a family of positive superharmonic functions. The inequalities (4.6) and (4.7) reflect the continuity of $u$ in the fine topology.

We have already noted the interpretation of theorem 4.1 in terms of downcrossings and upcrossings. This interpretation suggests the following theorem.

THEOREM 4.2. *Let* $\{y_j, \mathfrak{F}_j, j \leqq n\}$ *be a positive supermartingale, and let* $\gamma, \gamma'$ *be respectively the number of downcrossings and upcrossings of* $[s_1, s_2]$ *by* $y_1, \cdots, y_n$. *Then* (3.6) *and*

$$E\{\gamma' | \mathfrak{F}_1\} \leqq \frac{\min [y_1, s_1]}{s_2 - s_1} \tag{4.11}$$

*are true with probability* 1.

We have already proved (3.6). The only reason we restate it here is to stress that both it and its dual (4.11) can be proved by the techniques adapted from potential theory. In fact the proof we have given of theorem 4.1 can be interpreted as a proof of theorem 4.2 if we use the probabilistic counterparts of the potential-theoretic concepts of that proof. To do this we now define $A = [0, s_1]$ and $B = [s_2, \infty]$, when, as we have seen, the sums on the left in (4.6) and (4.7) in their probability interpretation, at the parameter value 1, become the conditional expectations on the left in (3.6) and (4.10). The quantities on the right in (4.6) and (4.7) become those on the right in (3.6) and (4.11).

## REFERENCES


[1] Y. S. CHOW, "Martingales in a $\sigma$-finite measure space indexed by directed sets," *Trans. Amer. Math. Soc.*, Vol. 97 (1960), pp. 254–285.

[2] J. L. DOOB, "Regularity properties of certain families of chance variables," *Trans. Amer. Math. Soc.*, Vol. 47 (1940), pp. 455–486.

[3] ———, *Stochastic Processes*, New York, Wiley, 1953.

[4] G. E. HUNT, "Markoff chains and Martin boundaries," *Illinois J. Math.*, Vol. 4 (1960), pp. 313–340.

[5] J. L. SNELL, "Application of martingale system theorems," *Trans. Amer. Math. Soc.*, Vol. 73 (1952), pp. 293–312.